E2.5 Signals & Linear Systems.

Tutorial Sheet 5 – Solutions

1). a). $\mathcal{L}\{u(t)\} = \frac{1}{s}$ $\quad \mathcal{L}\left\{\frac{d}{dt}\right\} = s$.

Given the initial condition,

$$(s^2 + 3s + 2)\,Y(s) = s\left(\frac{1}{s}\right)$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}.$$

$$\therefore \quad y(t) = (e^{-t} - e^{-2t})\,u(t) \quad //$$

b) $(s^2 Y(s) - 2s - 1) + 4(sY(s) - 2) + 4Y(s)$

$$= (s+1) \times \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 4s + 4)\,Y(s) = 2s + 10$$

$$\Rightarrow \quad Y(s) = \frac{2s+10}{s^2+4s+4} = \frac{2s+10}{(s+2)^2} = \frac{2}{s+2} + \frac{6}{(s+2)^2}$$

$$\therefore \quad y(t) = (2 + 6t)\,e^{-2t}u(t) \quad //$$

c) $(s^2 Y(s) - s - 1) + 6(sY(s) - 1) + 25\,Y(s)$

$$= (s+2)\,\frac{25}{s} = 25 + \frac{50}{s}$$

$$\Rightarrow (s^2 + 6s + 25)\,Y(s) = \frac{s^2 + 32s + 50}{s}$$

$$\Rightarrow \quad Y(s) = \frac{s^2 + 32s + 50}{s(s^2 + 6s + 25)} = \frac{2}{s} + \frac{-s + 20}{s^2 + 6s + 25}$$

$$\therefore \; y(t) = \left[2 + 5.836\,e^{-3t}\cos(4t - 99.86^\circ)\right] u(t) \quad //$$

2) a) $$Y(s) = \frac{5s+3}{s^2+11s+24}$$ // (Write down directly!)

b) $$Y(s) = \frac{3s^2+7s+5}{s^3+6s^2+11s+6}$$ //

c) $$Y(s) = \frac{3s+2}{s^4+4s}$$

$$= \frac{3s+2}{s(s^3+4)}$$ //

3) a) i) $f(t) = e^{-4t}u(t) \quad \therefore \quad F(s) = \frac{1}{s+4}$

$$Y(s) = H(s)\,F(s) = \frac{s+5}{(s+2)(s+3)(s+4)}$$

$$= \frac{3/2}{s+2} - \frac{2}{s+3} + \frac{1/2}{s+4}$$

$$\therefore \quad y(t) = \left[\frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t}\right]u(t)$$ //

(Easy with LT!)

ii) $f(t) = e^{-3t}u(t) \quad \therefore \quad F(s) = \frac{1}{s+3}$.

$$Y(s) = \frac{s+5}{(s+2)(s+3)^2} = \frac{3}{s+2} - \frac{3}{s+3} - \frac{2}{(s+3)^2}$$

$$\therefore \quad y(t) = \left(3e^{-2t} - 3e^{-3t} - 2te^{-3t}\right)u(t)$$ //

iii) $f(t) = e^{-4(t-5)} u(t-5)$

~~$\therefore F(s)$~~

Note that this i) delayed by 5 sec.

$$\therefore F(s) = \left(\frac{1}{s+4}\right) e^{-5s}$$ (Time shifting property)

$$Y(s) = \frac{s+5}{(s+2)(s+3)(s+4)} e^{-5s}$$

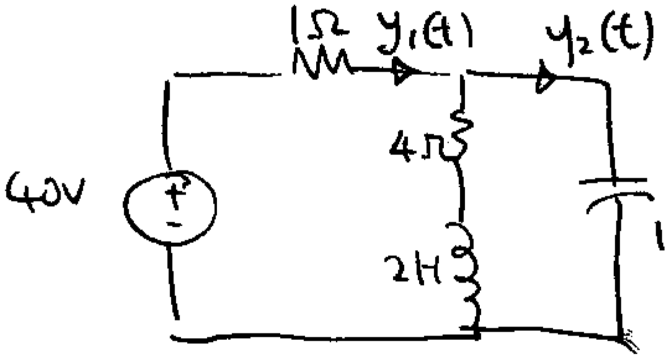
$$= \left[\frac{3/2}{s+2} - \frac{2}{s+3} + \frac{1/2}{s+4}\right] e^{-5s}$$

$$\therefore y(t) = \left[\frac{3}{2} e^{-2(t-5)} - 2e^{-3(t-5)} + \frac{1}{2} e^{-4(t-5)}\right] u(t-5)$$ //

b) $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y(t) = 2\frac{df}{dt} + 3f(t)$$ //

4. At $t=0$, inductor current $y_1(0)=4A$. (4)

Capacitor voltage is 16 V.



After $t=0$, loop equations are :— (Voltage around loops)

$$2\left(\frac{dy_1}{dt}-\frac{dy_2}{dt}\right)+y_1(t)+4\left(y_1(t)-y_2(t)\right)=40$$

$$\Rightarrow\quad 2\frac{dy_1}{dt}-2\frac{dy_2}{dt}+5y_1(t)-4y_2(t)=40 \qquad (1)$$

and

$$-2\frac{dy_1}{dt}-4y_1(t)+2\frac{dy_2}{dt}+4y_2(t)+\int_{-\infty}^{t}y_2(\tau)\,d\tau=0 \qquad (2)$$

Initial conditions gives us:

$$y_1(t)\Longleftrightarrow Y_1(s),\quad \frac{dy_1}{dt}=sY_1(s)-4$$

$$y_2(t)\Longleftrightarrow Y_2(s),\quad \frac{dy_2}{dt}=sY_2(s)$$

$$\int_{-\infty}^{t}y_2(\tau)\,d\tau\Longleftrightarrow\frac{1}{s}Y_2(s)+\frac{16}{s}$$

Laplace Transform of the loop equations are therefore:

$$2(sY_1(s)-4)-2sY_2(s)+5Y_1(s)-4Y_2(s)=\frac{40}{s}$$

$$\Rightarrow\quad (2s+5)\,Y_1(s)-(2s+4)\,Y_2(s)=8+\frac{40}{s} \qquad (1s)$$

$$-2(sY_1(s)-4)-4Y_1(s)+2sY_2(s)+4Y_2(s)+\frac{1}{s}Y_2(s)+\frac{16}{s}=0$$

$$\Rightarrow\quad -(2s+4)Y_1(s)+\left(2s+4+\frac{1}{s}\right)Y_2(s)=-8-\frac{16}{s}$$

(2s)

Write in matrix form:

$$\begin{bmatrix} (2s+5) & -(2s+4) \\ -(2s+4) & \left(2s+4+\frac{1}{s}\right) \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} 8+\frac{40}{s} \\ -8-\frac{16}{s} \end{bmatrix}$$

Apply Cramer's rule yields

$$Y_1(s) = \frac{4(6s^2+13s+5)}{s(s^2+3s+2.5)} = \frac{8}{s} + \frac{16s+28}{s^2+3s+2.5}$$

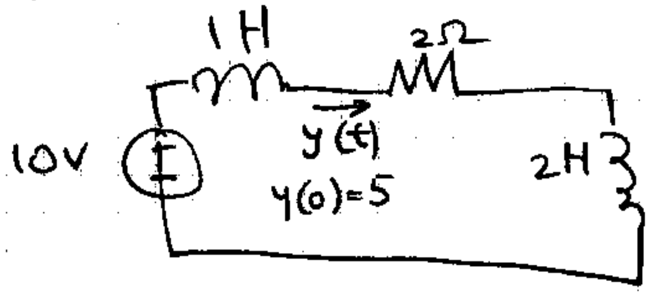
Use table:

$$\therefore \quad y_1(t) = \left[8 + 17.89\, e^{-1.5t} \cos\left(\frac{t}{2} - 26.56^\circ\right)\right] u(t)$$

$$Y_2(s) = \frac{20(s+2)}{(s^2+3s+2.5)}$$

$$\therefore \quad y_2(t) = 20\sqrt{2}\, e^{-1.5t} \cos\left(\frac{t}{2} - \frac{\pi}{4}\right) u(t)$$

5. Before switch is opened, (ie $t=0$) inductor current is 5A, ie. $y(0)=5$.



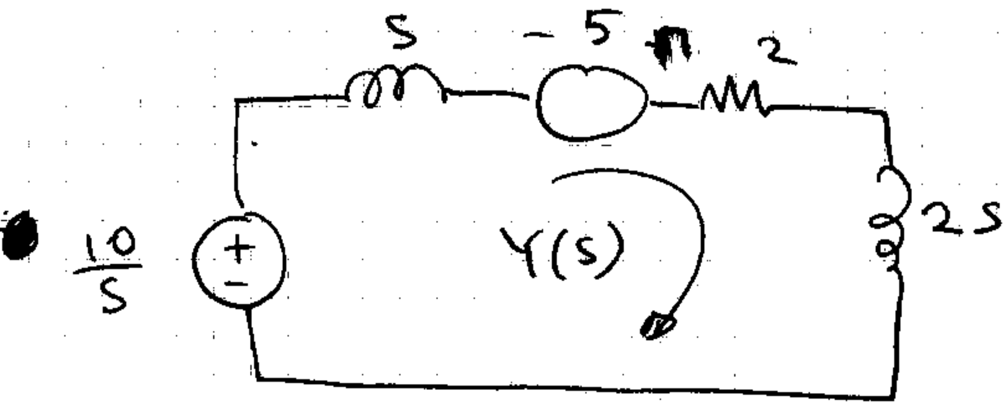
$$Y(s) = \frac{10/s + 5}{3s+2}$$

$$= \frac{5s+10}{s(3s+2)}$$

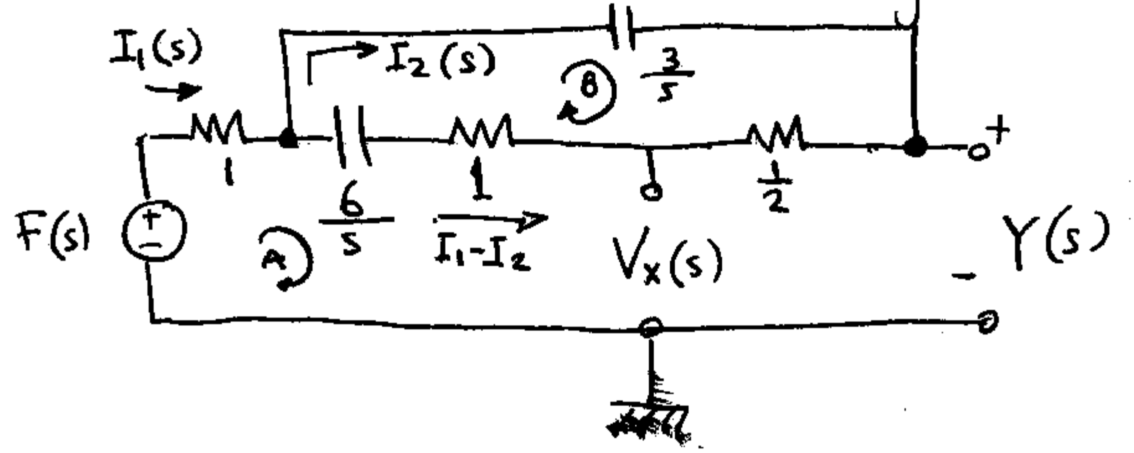
$$= \frac{5}{3}\left[\frac{3}{s} - \frac{2}{s+\frac{2}{3}}\right]$$

$$\therefore\ y(t) = \left(5 - \frac{10}{3}e^{-\frac{2}{3}t}\right)u(t)$$

Transformed circuit in s-domain:



6. The transformed circuit for Fig Q6 :



Since op amp has infinite gain,

$V_x(s) \simeq 0$.

loop equations are : —

(A) $1 \times I_1(s) + \left(\frac{6}{s} + 1\right)\left[I_1(s) - I_2(s)\right] = F(s)$

(B) $-\frac{3}{2} I_1(s) + \left(\frac{6}{s} + \frac{3}{2}\right)\left[I_1(s) - I_2(s)\right] = 0$

$$\begin{bmatrix} \left(\frac{6}{s}+2\right) & -\left(\frac{6}{s}+1\right) \\ \frac{6}{s} & \left(\frac{6}{s}+\frac{3}{2}\right) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

Cramer's rule yields

$$I_1(s) = \frac{s(s+6)}{s^2+8s+12} F(s) , \qquad I_2(s) = \frac{s(s+4)}{s^2+8s+12} F(s)$$

$$Y(s) = -\frac{1}{2} \times \left[I_1(s) - I_2(s)\right] .$$

$$= \frac{-s}{s^2+8s+12} F(s)$$

$\therefore$ Transfer Function

$$H(s) = \frac{Y(s)}{F(s)} = -\frac{s}{s^2+8s+12} \quad //$$

7) (a) $u(t) \iff \frac{1}{s}$

$$Y(s) = \frac{6s^2+3s+10}{s(2s^2+6s+5)}$$

$$\therefore y(0^+) = \lim_{s\to\infty} sY(s) = 3$$

$$y(\infty) = \lim_{s\to 0} sY(s) = 2 \quad //$$

(b) $e^{-t}u(t) \iff \frac{1}{s+1}$

$$\therefore Y(s) = \frac{6s^2+3s+10}{(s+1)(2s^2+6s+5)}$$

$$\therefore y(0^+) = \lim_{s\to\infty} sY(s) = 3$$

$$y(\infty) = \lim_{s\to 0} sY(s) = 0 \quad //$$

8. $$H(s) = \frac{s+3}{(s+2)^2}$$

$$\therefore H(j\omega) = \frac{j\omega+3}{(j\omega+2)^2}$$

$$|H(j\omega)| = \frac{\sqrt{\omega^2+9}}{\omega^2+4} \qquad \angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{3}\right) - \tan^{-1}\left(\frac{\omega}{2}\right)\times 2$$

(a) $f(t) = \cos(2t+60^\circ) \quad \therefore \omega = 2$

$$|H(j2)| = \frac{\sqrt{13}}{8} \quad \text{and} \quad \angle H(j2) = 33.69^\circ - 2\times 45^\circ$$
$$= -56.31^\circ.$$

$$\therefore y(t) = \frac{\sqrt{13}}{8}\cos(2t+60^\circ-56.31^\circ)$$

$$= \frac{\sqrt{13}}{8}\cos(2t+3.69^\circ) \quad //$$

8 (b) $f(t) = \sin(3t - 45°)$. Hence $\omega = 3$,

$|H(j3)| = \frac{\sqrt{18}}{13}$ and $\angle H(j3) = 45° - 112.62°$
$= -67.62°$.

$\therefore\ y(t) = \frac{\sqrt{18}}{13} \sin(3t - 112.62°)$ //

(c) $f(t) = e^{j3t}$ $\qquad w = 3$

$y(t) = H(jw) e^{j3t} = H(j3) e^{j3t}$

$= |H(j3)|\, e^{j(3t + \angle H(j3))}$ From (b)

$= \frac{\sqrt{18}}{13} \sin(3t - 67.62°)$ //

10. a) If r and d are the distance of the zero and pole from $j\omega$ respectively, $|H(j\omega)|$ is the ratio $\frac{r}{d}$ corresponding to $j\omega$.

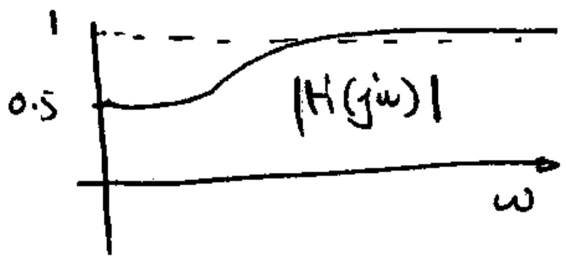
At $\omega = 0$, $\frac{r}{d} = 0.5$

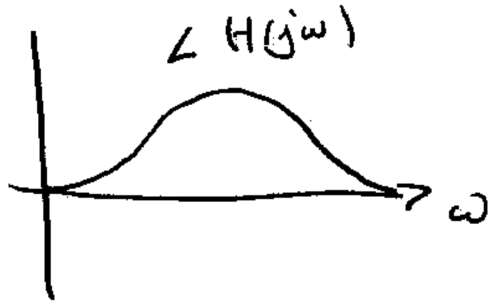
$\omega = \infty$, $\frac{r}{d} = 1$.

~~$\angle H(j\omega)$~~

At $\omega = 0$, $\angle H(j\omega) = 0$
$\omega \to \infty$, $\angle H(j\omega) = 0$.

In between, the angle is +ve.





b)